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# A FINITE ELEMENT APPROXIMATION OF NONLINEAR FLOW IN POROUS MEDIA

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## ABSTRACT

For high flow velocities through porous materials Darcy's law no longer describes the relationship between the hydraulic gradient and velocity, thus more general expressions are required. In this work we consider an expression of exponential type to formulate the steady-state nonlinear flow problem. The - dependence of the media permeability with the hydraulic gradient suggest the use of the Lagrangien augmented methods. Here we present an algorithm based both in these methods and in a - discretitation of the flow field by finite elements wich take into account the possible apparition of a free surface. Finally, we apply this method to the calculation of an unconfined non-Darcy flow near an open mine where a system of wells are - perforated to avoid the flooding of the mine.

## INTRODUCTION

The finite element method has been extensively applied - to the solution of linear flow through porous media both in -- the steady-state case by Zienkiewicz, Mayer and Cheung (1966) or the evolution case as in Zienkiewicz and Parekh (1970)or -- Pinder and Frind (1972) which combine different types of finite elements. In the other hand the first algorithms to the - - treatement of the free surface were based in an iterative modification of the finite element mesh as in Neuman and Witherspoon (1971) or Frang, Wang and Harrison (1972); Later Bathe -- and Koshgoftaar (1979) have proposed a method which do not require mesh iteration. Recently Desai (1984) used a refinement of the Bathe's method which take into account the non saturated area. All the preceding works concern to linear flow - - - through porous media.When high velocities take place the Darcy's equation does not describe correctly the relationship between the hydraulic gradient and velocity, thus general expressions has been proposed as:

$$i = c\, v^{m} \tag{1}$$

$$i = av + bv^{2} \tag{2}$$

where i is the hydraulic gradient and v is the velocity of the fluid, a and b are constants of the porous media, however both c and m depend of the velocity. The value of m ranges from one, in the linear case, to two for turbulent flow. In Ahmed and Sunada (1969) we can find a theoretical justification to equation (2) known as Forchheimer equation. In the other hand Pérez Franco (1982) proves that when the hydraulic gradient variations in the flow field are relatively small it is possible to consider c and m as constant values. General studies has been made using the former expressions and the finite element method as in Mc Corquodale (1970) which used the equation (2) and an over relaxation method to solve the system of equations; Volker (1969) used both expressions combined with the method proposed by Finn (1967) for the treatement of the free surface. In this work we adopt the exponential expression (1) to formulate the nonlinear flow in porous media problem and an Lagrangien augmented method which allows to uncoupling the nonlinear problem in each point; reversing the expression (1) we have, with $i = |\vec{\nabla} u|$, u denoting the piezometric head:

$$\vec{v} = k_n\, |\vec{\nabla} u|^{n-1}\, \vec{\nabla} u \tag{3}$$

where $|.|$ denote the vector norm in $R^d$ (d=1,2 or3) and $k_n$ and n its depends of the $|\vec{\nabla} u|$; in the linear case we have n=1 and in the turbulent case n=0.5. The steady-state flow equations in a isotropic media will be:

$$-\nabla(k_n |\vec{\nabla} u|^{n-1})\, \vec{\nabla} u = f \quad \text{in} \quad \Omega \tag{4}$$

$$u = u_0 \quad \text{in} \quad \Gamma_0 \tag{5}$$

$$k_n |\vec{\nabla} u|^{n-1} \frac{\partial u}{\partial n} = g \quad \text{in} \quad \Gamma_1 \tag{6}$$

where $\Omega \subset R^d$ is the domain occupied by the porous media, $\Gamma = \Gamma_0 \cup \Gamma_1$ is the boundary of $\Omega$ and f, $u_0$ and g are known functions which represent the flow from external sources, the piezometric head on $\Gamma_0$ and the flow through $\Gamma_1$.

## VARIATIONAL FORMULATION

The stated problem (4), (5), (6) can be formulated as an optimization one in the following way (see Ciarlet (1978)):

$$J(u) = \text{Min}\ J(v) \tag{7}$$

$$J(v) = \frac{1}{s}\int_\Omega K_n|\vec{\nabla} v|^s dx - \int_\Omega fvdx - \int_{\Gamma_1} gvdx \qquad (8)$$

where s=n+1, and we search the minimum between the set of functions which takes the value $u_0$ on the boundary $\Gamma_0$.

Safe for the case s=2 (Darcy's law) the former problem - is nonlinear because the relationship of the permeability and the gradient of the solution. This kind of dependence suggest the introduction of a new variable and a new equation $\vec{p} = \vec{\nabla}u$ which allows to uncouple the difficulties besides the gradient and in the other hand the nonlinear character of the material. The expressions (7) and (8) should be transformed into the folowwing:

$$J(u,\vec{p}) = \text{Min } J(v,\vec{q}) \qquad (9)$$

$$J(v,\vec{q}) = \frac{1}{s}\int_\Omega k_n|\vec{q}|^s dx - \int_\Omega fvdx - \int_{\Gamma_1} gvdx \qquad (10)$$

and now the minimum is search between the couples $(v,\vec{q})$ which verify the relations $v|_{\Gamma_0} = u_0$ and $\vec{q} = \vec{\nabla}v$

## FINITE ELEMENT APPROXIMATION

In practice we solve a discretized version by finite elements of the minimization problem (9)-(10); considering the domain divided into triangular elements and taking continous - functions in $\Omega$ which are a polynome of degree k in each triangle; in the other hand we take each component in $\vec{q}=\{q_1,q_2\}$ as a polynome of degree k-1 in each element but without any continuity relation among elements.

## NUMERICAL ALGORITHM

One method to solve (9)-(10) is to hand the equation - - $\vec{q}=\vec{\nabla}v$ introducing a Lagrange multiplier $\vec{\mu}$ and to replace $J(v,\vec{q})$ by the Lagrangien function:

$$\mathcal{L}(v,\vec{q},\vec{\mu}) =$$
$$\frac{1}{s}\int_\Omega k_n|\vec{q}|^s dx - \int_\Omega fvdx - \int_{\Gamma_1} gvdx + \int_\Omega k_D\vec{\mu}(\vec{\nabla}v-\vec{q})dx \qquad (11)$$

where $k_D$ is the Darcy's permeability whose introduction here - we will justify later. It is easy to verify that the required solution is a saddle point of the Lagrangien function $\mathcal{L}$, that means, $(u,\vec{p},\vec{\lambda})$ is a solution of:

$$\mathcal{L}(u,\vec{p},\vec{\mu}) \leqslant \mathcal{L}(u,\vec{p},\vec{\lambda}) \leqslant \mathcal{L}(v,\vec{q},\vec{\lambda}) \qquad (12)$$

for each value $(v,\vec{q},\vec{\mu})$

From a numerical point of view the solution of (12) rise some dificulties because the bad conditioning of the system -- equations matrix;beside that Hestenes (1969) suggested the -- following modification of the Lagrangien function $\pounds$, with $r>0$:

$$\pounds_r(v,\vec{q},\vec{\mu}) = \pounds(v,\vec{q},\vec{\mu}) + \frac{r}{2}\int k_D|\vec{\nabla}v-\vec{q}|^2\,dx$$

and the solution is now a saddle point of $\pounds_r$.

The searching algoritm of a saddle point is a modifica-- tion of Uzawa's method and is studied in Fortin and Glowinski (1982):

We start with arbitrary values for $\vec{p}^{\circ}$ and $\vec{\lambda}^{\circ}$

a) Once $\vec{p}^{i}$ and $\vec{\lambda}^{i}$ are known, we calculate $u^{i+1}$ solving

$$\pounds_r(u^{i+1},\vec{p}^{i},\vec{\lambda}^{i}) = \operatorname*{Min}_{v}\ \pounds_r(v,\vec{p}^{i},\vec{\lambda}^{i}) \tag{13}$$

b) We calculate $\vec{p}^{i+1}$ solving

$$\pounds_r(u^{i+1},\ \vec{p}^{i+1},\lambda^{i}) = \operatorname*{Min}_{\vec{q}}\ \pounds_r(u^{i+1},\vec{q},\vec{\lambda}^{i}) \tag{14}$$

c) We calculate $\lambda^{i+1}$ by ($\rho>0$):

$$\vec{\lambda}^{i+1} = \vec{\lambda}^{i} + \rho(\vec{\nabla}u^{i+1}-\vec{p}^{i+1}) \tag{15}$$

If $|\vec{\nabla}u^{i+1}-\vec{p}^{i+1}| > \varepsilon\ |\vec{\nabla}u^{i+1}|$ make i equal to i+1 and go to a), in another case stop.

The step (13) is equivalent to solve the following li- - near problem for the variable $u^{i+1}$

$$r\int_\Omega k_D\vec{\nabla}u^{i+1}\vec{\nabla}v\,dx = \int_\Omega k_D(r\ \vec{p}^{i}-\vec{\lambda}^{i})\vec{\nabla}v\,dx + \int_\Omega fv\,dx + \int_{\Gamma_1} gv\,d\Gamma \tag{16}$$

The system equation matrix is fixed and consequently only one factorization is required.

The problem (14) is solved in each integration point be- cause the variable $\vec{q}$ is not continous; in each point we have to solve the system of d equations:

$$k_n|\vec{p}^{i+1}|^{s-2}\,\vec{p}^{i+1} + r\ k_D\ \vec{p}^{i+1} = r(\vec{\nabla}u^{i+1}+\lambda^{i})k_D \tag{17}$$

In practice the nonlinear problem can be reduced to only one equation, if the value $z = |\vec{p}^{i+1}|$ is known, then the vector $\vec{p}^{i+1}$ can be calculated explicitly from (17); z is obtained sol- ving by Newton's method:

$$k_n\ z^{s-1} + r\ k_D\ z = k_D\ w \tag{18}$$

where $w = |r\, \vec{\nabla} u^{i+1} + \vec{\lambda}^{i}|$

Finally we observe that (15) is an explicit computation en each integration point.

The introduction of Darcy's permeability allows the algorithm to solve linear problems in which case we have convergence for the variable u in only one iteration if we select $\rho = r = T$. (See Glowinski and Marroco (1975)).

When the values $k_n$ and s change with $|\vec{\nabla} u|$, we modify the step b) in the following way:

b1) Compute $\vec{\nabla} u^{i+1}$

b2) Compute $k_n = k_n(\vec{\nabla} u^{i+1})$, $s = s(\vec{\nabla} u^{i+1})$

b3) Solve (17)

## TREATMENT OF THE FREE SURFACE

The method used for handling the free surface in unconfined aquifers is similar to that outlined by Bathe and Khosgoftaar (1979) for the linear case (see also Desai (1984); they introduce a definition of k as follows:

$$k(u) = \begin{cases} k_D & \text{if } u > y \\ 0 & \text{if } u \leqslant y \end{cases}$$

where y is the geometric head of the point; then a modified Newton's procedure is adopted. In the nonlinear case we introduce the same corrector term in the expression (16) and we obtain:

$$r \int_{\Omega} k_D\, \vec{\nabla}\delta \vec{\nabla} v\, dx = \int_{\Omega} k(u_i)(r\vec{p}^{i} - \vec{\lambda}^{i})\vec{\nabla} v\, dx + \int_{\Omega} f v\, dx + \int_{\Gamma_1} g v\, d\Gamma - r\int_{\Omega} k(u^{i})\vec{\nabla} u^{i} \vec{\nabla} v\, dx$$

and $u^{i+1} = u^{i} + w\delta$, where w is an acceleration parameter ($w \geqslant 1$).

## COMPUTATION OF THE FLOW THROUGH A SURFACE

For the computation of the flow that run through a known surface we use the nodal reaction concept (see Zienkiewicz (1980); as indicated by Hinton and Owen (1979) the flow going across a boundary where we know the piezometric head, can be computed by addition of the nodal reactions in the nodes of the boundary; denoting by $\Psi_I$ the base function associated to the node I of the $\Gamma_0$ boundary, we have from (14)

$$\int_{\Omega} k_n |\vec{\nabla} u|^{n-1}\, \vec{\nabla} u \vec{\nabla} \Psi_I\, dx - \int_{\Gamma_0} k_n |\vec{\nabla} u| \frac{\partial u}{\partial n} \Psi_I\, d\Gamma = \int_{\Omega} f \Psi_I\, dx + \int_{\Gamma_1} g\, \Psi_I\, d\Gamma \qquad (20)$$

The integral term on $\Gamma_0$ represents the flow running through $\Gamma_0$ associated to node I and his value is

$$\phi_I = \int_\Omega k_n |\vec{\nabla} u|^{n-1} \vec{\nabla} u \vec{\nabla} \Psi_I dx - \int_\Omega f \Psi_I dx - \int_{\Gamma_1} g \Psi_I d\Gamma \qquad (21)$$

Usually the terms with f and g will be zero (zero) and the value $\Psi_I$ can be obtained from our algorithme after convergence ($i \to \infty$); we have from expression (20) and if we notice that $k(u)=k_D$ under the free surface:

$$\phi_I = \int_\Omega k_D (r\, \vec{p}^{\infty} - \vec{\lambda}^{\infty}) \nabla \vec{\Psi}_I dx + r \int_\Omega k_D \vec{\nabla} u^{\infty} \vec{\nabla} \Psi_I dx = \int_\Omega k_D\, \vec{\lambda}^{\infty} \vec{\nabla} \Psi_I dx \quad (22)$$

## APPLICATION

The efficience of the described algorithm has been proved in Ferragut and Elorza (1985) with several applications and compared the results with experimental data. Here we present an example of an open lengthy shape mine with a system of wells parallels to the longest axis to avoid the flooding of the mine. First in a middle section the hydrological conditions are thus indicated in figure 1 where we have designed with pointed-line the future position of the wells. The boundary value problem, taking into account symmetry is showed in figure 2. The computations of the nodal reactions in the well should allow us to know the necessary flow to maintain the free surface under the surface of the mine.

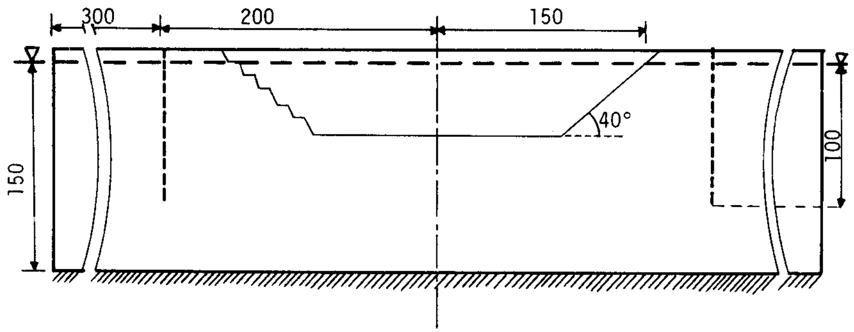


Figure 1. Middle section

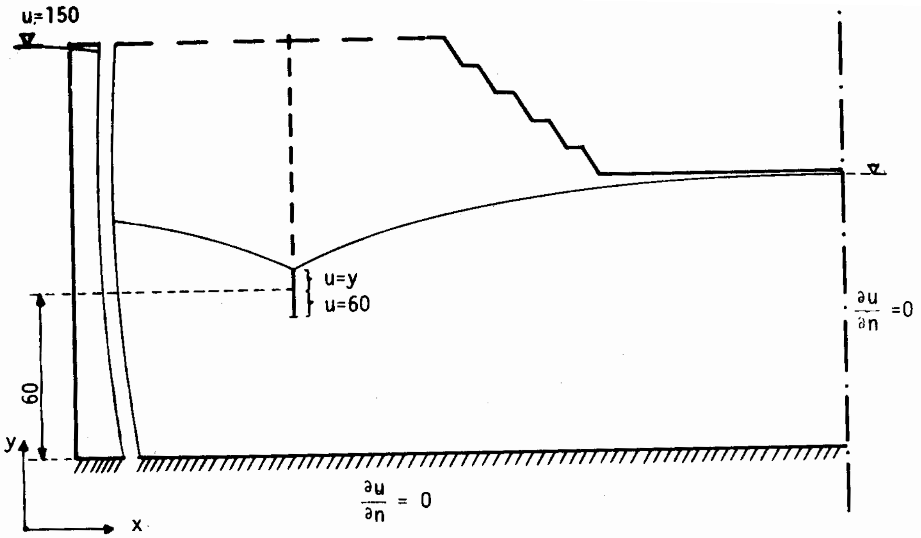


Figure 2. Boundary conditions

We have considered nonlinear flow in a karstic rock with a Darcy permeability $k_D$=2000 m/d and a ratio from turbulent permeability and Darcy's one of $k_T/k_D$ = 0.5; Hernández Valdés (1981) describes a method to compute from the rario $k_T/k_D$ the n value in (3) and the ratio $k_n/k_D$, so table 1 is obtained.

Table 1. n and $k_n/k_D$ values

| $k_n$ | $k_n/k_D$ | n | $\lvert\vec{\nabla} u\rvert$ |
|---|---|---|---|
| 2000. | 1. | 1. | $10^{-4}-10^{-2}$ |
| 1200. | 0.6 | 0.89 | $10^{-2}-10^{-1}$ |
| 580. | 0.29 | 0.69 | $10^{-1}-10^{0}$ |
| 520. | 0.26 | 0.56 | $10^{0}-10^{1}$ |
| 580. | 0.29 | 0.52 | $10^{1}-10^{2}$ |

We have done two finite element models; the first one -- with 568 nodes and 1008 triangles, as can be seen in figure 3 and a second model with 1022 nodes and 1886 triangles (see figure 4) where we have avoided partly the dry zone and a mesh refinement has been considered. In figures 3 and 4 we present as well the position of the free surface obtained with both models. In table 2 we give the nodal reactions, for the second model, corresponding to the nodes on the boundary u=150; the addition equal to 2600 $m^3/d$ is the necessary flow to maintain the free surface in the required position.

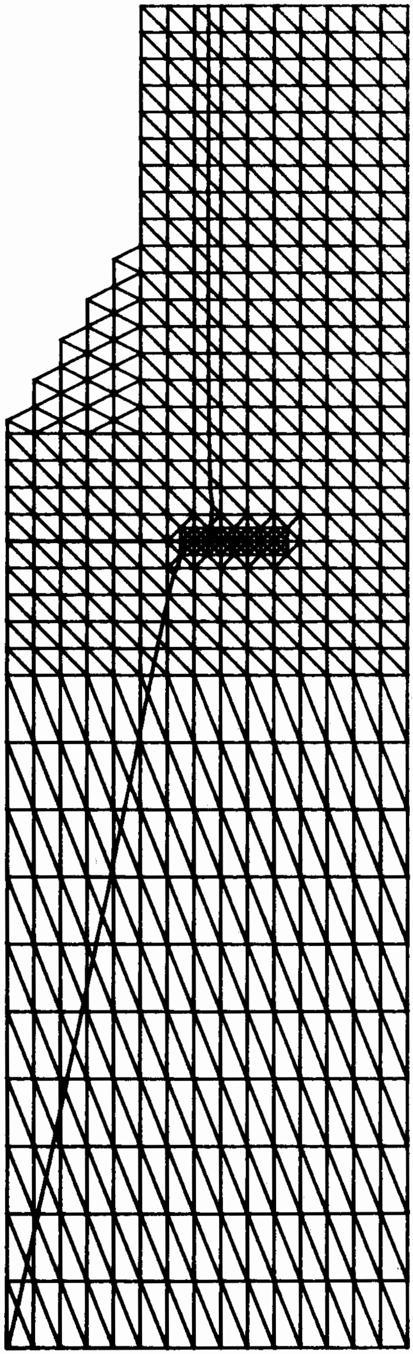

Figure 3. Model-1: Mesh and free surface

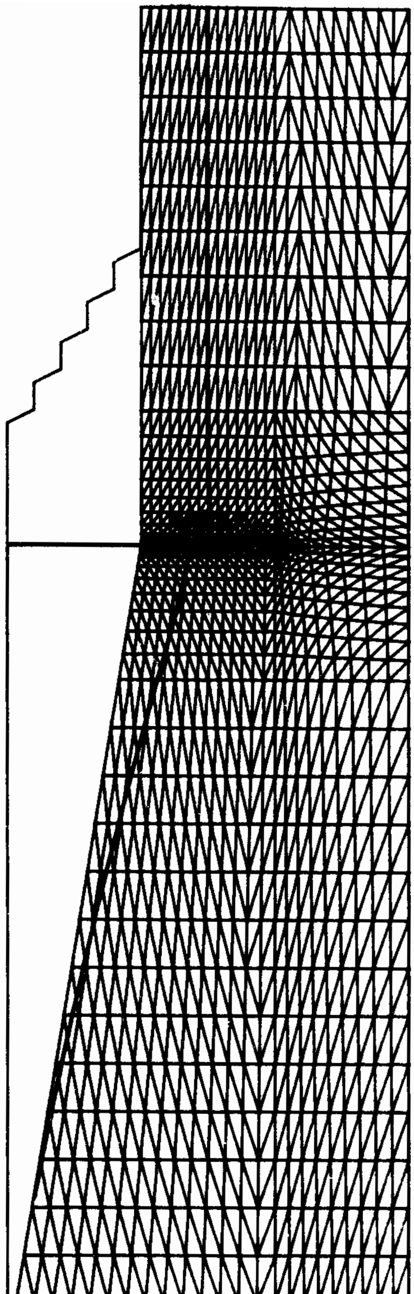

Figure 4. Model-2: Mesh and free surface

Table 2. Nodal reactions in u=150.(x $10^3$)

| NODE | N.R. | NODE | N.R. | NODE | N.R. | NODE | N.R. | NODE | N.R. | NODE | N.R. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.31 | 61 | 1.20 | 121 | 1.28 | 181 | 1.32 | 241 | 1.04 | 301 | 1.05 |
| 16 | 1.10 | 76 | 1.23 | 136 | 1.29 | 196 | 1.33 | 256 | 1.05 | 316 | 1.05 |
| 31 | 1.14 | 91 | 1.25 | 151 | 1.30 | 211 | 1.18 | 271 | 1.05 | 331 | 1.05 |
| 46 | 1.17 | 106 | 1.26 | 166 | 1.31 | 226 | 1.04 | 286 | 1.05 | 346 | 0.52 |

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